

# Q-lumps on domain wall with spin-orbit interaction

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## Abstract

The nonlinear  $O(3)$   $\sigma$ -model in (2+1) dimensions with an additional potential term admits solutions called Q-lumps, having both topological and Noether charges. We consider in 3+1-dimensional spacetime the theory with Q-lumps on a domain wall in presence of spin-orbit interaction in bulk and find interaction effects for two-particle solution through perturbation theory and adiabatic approximation.

*Keywords:* Q-lumps,  $O(3)$   $\sigma$ -model, domain wall, spin-orbit interaction

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## 1. Introduction

Topological defects occur in many topics of field theory, cosmology and condensed matter physics. There is a particular interest to study models with additional nonabelian internal degrees of freedom localized on solitons (domain walls, strings, monopoles etc.), see [1–9]. The next possible step is to consider theories containing solitons on solitons (as in [10]). In this paper we study Q-lumps on domain wall (see similar construction in [11], full numerical solution in [12] and generalization to higher dimension in [13]).

Q-lumps are Q-ball type solutions of nonlinear  $O(3)$   $\sigma$ -model with an additional potential term of some special form. They were discovered by R. Leese [14] and have nontrivial topological charge in addition to the nonzero conserved Noether charges. Properties of such configurations differ significantly from those of pure  $\sigma$ -model solution even when coupling constant is small, e.g. in case of small perturbation to the action, that may be important from physical point of view as every physical system is unlikely to contain no perturbation at all. The main result of R. Leese was that he first managed to obtain explicit solutions to  $O(3)$   $\sigma$ -model with a particular potential term and to investigate stability and scattering effects of them, later the mechanism of building of such configurations was generalized to arbitrary  $\sigma$ -models by E. Abraham [15].

One special feature of Q-lumps is that the model admits stationary many-soliton solutions that can be interpreted as non-interacting particles (relevant to initially motionless configuration). In this paper we investigate whether and what interaction appears if one adds spin-orbit interaction term in the bulk (as in [5],[6],[8]).

The main body of the paper looks at the interaction of two Q-lumps on a domain wall. Sect. 2 presents the model we deal with and introduces all the objects (the wall, Q-lumps, spin-orbit interaction term) in details. Sect. 3 investigates interaction effects perturbatively and through adiabatic approximation, finally, effects found are outlined in sect. 4.

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## 2. Model

### 2.1. Action

The theory considered includes scalar fields  $\phi \in \mathfrak{R}$  and  $\chi_i \in \mathfrak{R}$ ,  $i = 1, 2, 3$  in 3+1-dimensional spacetime and admits domain wall with additional nonabelian internal degrees of freedom localized on it. Lagrangian of the theory is

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\chi, \quad (2.1)$$

$$\mathcal{L}_\phi = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \lambda(\phi^2 - v^2)^2, \quad (2.2)$$

$$\mathcal{L}_\chi = \frac{1}{2} \partial_\mu \chi_i \partial^\mu \chi_i - \gamma((\phi^2 - \mu^2)\chi_i \chi_i + \beta(\chi_i \chi_i)^2) - \frac{1}{2} \alpha^2(\chi_i \chi_i - \chi_3 \chi_3). \quad (2.3)$$

To build a domain wall let us look for solutions with separated variables  $\phi = \phi(x_3)$ ,  $\chi_i = \chi(x_3) S_i(t, x, y)$ , where  $S_i S_i = 1$ , and fields  $S_i$  correspond to O(3)  $\sigma$ -model. Thus the Lagrangian takes form

$$\begin{aligned} \mathcal{L} = \mathcal{L}_\phi + \frac{1}{2} \chi^2 \partial_p S_i \partial^p S_i - \frac{1}{2} (\chi')^2 - \gamma((\phi^2 - \mu^2)\chi^2 + \beta\chi^4) - \frac{1}{2} \alpha^2 \chi^2 (1 - S_3^2) = \\ = \mathcal{L}_{wall} + 2\chi^2 \mathcal{L}_{lump}, \end{aligned} \quad (2.4)$$

where  $p = 0, 1, 2$ , differentiation over  $x_3$  is denoted by prime and

$$\mathcal{L}_{lump} = \frac{1}{4} \partial_p S_i \partial^p S_i - \frac{1}{4} \alpha^2 (1 - S_3^2). \quad (2.5)$$

Motion equations derived from action (2.1) are:

$$\phi'' = 4\lambda(\phi^2 - v^2)\phi + 2\gamma\chi^2\phi, \quad (2.6)$$

$$\chi'' - 2\gamma(\phi^2 - \mu^2)\chi - 4\beta\gamma\chi^3 + 4\chi\mathcal{L}_{lump} = 0, \quad (2.7)$$

$$\partial_p \partial^p u - \frac{2u^* \partial_p u \partial^p u}{1 + uu^*} + \frac{\alpha^2 u (1 - uu^*)}{1 + uu^*} = 0. \quad (2.8)$$

### 2.2. Q-lumps

The model with Lagrangian (2.5) in three-dimensional spacetime was studied in detail by Robert Leese in paper [14]: he managed to find explicit form of solitonic solutions having both topological and Noether charges related respectively to homotopic group  $\pi_2(S^2) = \mathbb{Z}$  and to internal rotation with constant velocity around  $S_3$  (one can notice that potential term breaks O(3) symmetry of the model to O(2)). These solitons were named "Q-lumps" (by analogy with S. Coleman's "Q-balls" [16]). They are not forbidden by Derrick's theorem (e.g. [17]) as they are not static and are stabilized by charge. Let's write them and their properties explicitly.

For convenience we introduce complex scalar fields  $u$ ,  $u^*$  instead of  $S_i$  through stereographic projection:  $u = \frac{S_1 + iS_2}{1 - S_3}$ , then

$$\mathcal{L}_{lump} = \frac{\partial_p u \partial^p u^*}{(1 + uu^*)^2} - \frac{\alpha^2 uu^*}{(1 + uu^*)^2}. \quad (2.9)$$

Topological and Noether charges and total energy of configuration look like

$$N = \frac{1}{4\pi} \int \vec{S} \cdot [\partial_x \vec{S} \times \partial_y \vec{S}] d^2x = \frac{i}{2\pi} \int \frac{\partial_x u^* \partial_y u - \partial_x u \partial_y u^*}{(1 + uu^*)^2} d^2x, \quad (2.10)$$

$$Q = \int \frac{1}{2} (S_2 \partial_t S_1 - S_1 \partial_t S_2) d^2x = i \int \frac{u^* \partial_t u - u \partial_t u^*}{(1 + uu^*)^2} d^2x, \quad (2.11)$$

$$\mathcal{E}_{lump} = \int \frac{\partial_t u \partial_t u^* + \partial_i u \partial_i u^* + \alpha^2 uu^*}{(1 + uu^*)^2} d^2x \quad (2.12)$$

and are linked by Bogomolny bound

$$\mathcal{E}_{lump} \geq 2\pi |N| + |\alpha Q|. \quad (2.13)$$

Thus configurations saturating Bogomolny bound have minimal energy among solutions with given charges and are classically stable because of charge conservation. Conditions of saturations are:

$$\partial_i u \pm i\varepsilon_{ij} \partial_j u = 0 \quad \text{and} \quad \partial_t u \pm i\alpha u = 0, \quad (2.14)$$

and that immediately gives explicit form of solutions:

$$u(t, x, y) = u_0(x, y) e^{\pm i\alpha t}, \quad u_0(x, y) = u_0(x \pm iy), \quad (2.15)$$

here function  $u_0(z = x + iy)$  must be (anti-)rational for energy to be finite. In that case degree of the function gives topological charge  $N$  of configuration, whereas Noether charge  $Q$  takes finite value when  $|N| \geq 2$ .

For instance, ansatz  $u(t, z = x + iy) = \left(\frac{\lambda}{z}\right)^k e^{i\alpha t}$  corresponds to radially symmetric soliton of topological charge  $k$ , while  $u(t, z) = \frac{\beta z + \gamma}{z^2 + \delta z + \epsilon} e^{i\alpha t}$  includes all solutions with top. charge 2. Configurations having several distant poles could be interpret as manyparticle solutions and one can study scattering processes on moduli space in the limit of low energies, e.g. papers [19], [20] considering scattering of monopoles and vortices, and in particular [21] and [22] about solitons of O(3)  $\sigma$ -model.

### 2.3. Domain wall

Term  $\mathcal{L}_{wall} = -\frac{1}{2}(\phi')^2 - \lambda(\phi^2 - v^2)^2 - \frac{1}{2}(\chi')^2 - \gamma((\phi^2 - \mu^2)\chi^2 + \beta\chi^4)$  allows to construct static domain wall of fields  $\phi(x_3)$  and  $\chi(x_3)$ : firstly one can see that in case of  $\chi(x_3) = 0$  motion equation (2.6) for field  $\phi(x_3)$  has ordinary kink solution  $\phi(x_3) = -v \tanh \frac{m_\phi}{2} (x_3 - x_{30})$ , but such a configuration is unstable ([5]). The stable one has nonzero expectation value  $\sqrt{\frac{\mu^2}{2\beta}}$  inside domain wall (i.e. around  $x_3 = x_{30}$ ). Profiles of functions  $\phi(x_3)$ ,  $\chi(x_3)$  shown in Fig. 2.1, 2.2 were derived numerically in the same paper [5] for some choice of parameters  $v, \lambda, \mu, \gamma, \beta$  of Lagrangian  $\mathcal{L}_{wall}$ .

Fields  $S_i$  like field  $\chi(x_3)$  become also localized on the wall around plane  $x_3 = x_{30}$  and in this way describe effectively two-dimensional (to be more precise, 2+1-dimensional) theory on domain wall.

### 2.4. Spin-orbit interaction

Now we add spin-orbit interaction in the bulk (for more detail about its origin see, for example, [18]). It breaks Lorentz-invariance of Lagrangian, leads to entanglement between fields  $\chi_i$  and coordinates and is effectively described by term

$$\mathcal{L}_{so} = -\varepsilon(\partial_i \chi_i)^2 \quad (2.16)$$

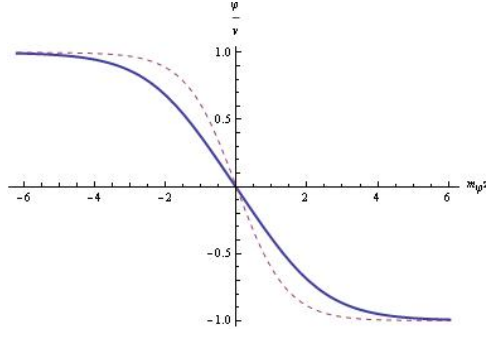


Figure 2.1: Numerical solution for  $\phi(z)$

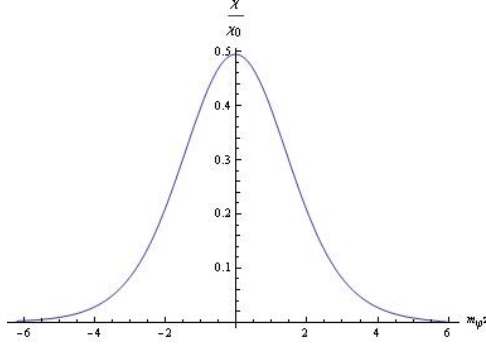


Figure 2.2: Numerical solution for  $\chi(z)$

or, in case of separated variables,  $\chi_i = \chi(z)S_i(t, x, y)$ ,

$$\mathcal{L}_{so} = -\varepsilon(\chi'^2(\frac{uu^* - 1}{1 + uu^*})^2 + 2\chi'\frac{uu^* - 1}{1 + uu^*}\chi\partial_k S_k + \chi^2(\partial_k S_k)^2), \quad (2.17)$$

where

$$\partial_k S_k = \frac{1}{(1 + uu^*)^2} (\partial_x u(1 - u^{*2}) + \partial_x u^*(1 - u^2) - i\partial_y u(1 + u^{*2}) + i\partial_y u^*(1 + u^2)). \quad (2.18)$$

To obtain effective action for theory on the wall in presence of this term one should integrate (2.1) + (2.17) over  $x_3$ . Then in 2+1 dimensions we get

$$\mathcal{L}_{eff} = A \left( \frac{\partial_p u \partial^p u^*}{(1 + uu^*)^2} - \frac{\alpha^2 uu^*}{(1 + uu^*)^2} \right) - \varepsilon \left( B(\frac{uu^* - 1}{1 + uu^*})^2 + C\frac{uu^* - 1}{1 + uu^*}\partial_k S_k + D(\partial_k S_k)^2 \right), \quad (2.19)$$

where constant values

$$A = \int 2\chi^2 dx_3, \quad (2.20)$$

$$B = \int (\chi')^2 dx_3, \quad (2.21)$$

$$C = \int 2\chi'\chi dx_3, \quad (2.22)$$

$$D = \int \chi^2 dx_3 = \frac{1}{2}A, \quad (2.23)$$

are introduced and in case of symmetric wall (function  $\chi(x_3)$ )

$$C = 0. \quad (2.24)$$

After division by a constant

$$\begin{aligned} \mathcal{L}_{eff} &= \frac{\partial_p u \partial^p u^*}{(1 + uu^*)^2} - \frac{\alpha^2 uu^*}{(1 + uu^*)^2} - \varepsilon \left( \frac{B}{A} \left( \frac{uu^* - 1}{1 + uu^*} \right)^2 + \frac{1}{2} (\partial_k S_k)^2 \right) = \\ &= \frac{1}{4} \partial_p S_i \partial^p S_i - \frac{1}{4} \alpha^2 (1 - S_3^2) - \varepsilon \left( \frac{B}{A} S_3^2 + \frac{1}{2} (\partial_k S_k)^2 \right) \end{aligned} \quad (2.25)$$

and up to a constant term

$$\begin{aligned} \mathcal{L}_{eff} &= \frac{1}{4} \partial_p S_i \partial^p S_i - \frac{1}{4} \alpha'^2 (1 - S_3^2) - \frac{\varepsilon}{2} (\partial_k S_k)^2 = \frac{\partial_p u \partial^p u^*}{(1 + uu^*)^2} - \frac{\alpha'^2 uu^*}{(1 + uu^*)^2} - \\ &\quad - \frac{\varepsilon (\partial_x u (1 - u^{*2}) + \partial_x u^* (1 - u^2) - i \partial_y u (1 + u^{*2}) + i \partial_y u^* (1 + u^2))^2}{2 (1 + uu^*)^4}. \end{aligned} \quad (2.26)$$

Thereby, of the three terms in (2.17) the first one leads to the correction of  $\alpha$

$$\alpha \rightarrow \alpha' = \sqrt{\alpha^2 - 4\varepsilon \frac{B}{A}}, \quad (2.27)$$

the second one vanishes due to the symmetry and only the third one can significantly influence the system.

### 3. Interaction

To find interaction effects let's consider two-particle solution of non-perturbated theory (without spin-orbit interaction). Two-particle configurations correspond to solutions of topological charge 2:

$$u_0(t, z = x + iy) = \frac{\gamma}{z^2 + \epsilon} e^{i\alpha't} = \frac{\gamma}{i\sqrt{\epsilon}} \left( \frac{1}{z - i\sqrt{\epsilon}} - \frac{1}{z + i\sqrt{\epsilon}} \right) e^{i\alpha't}. \quad (3.1)$$

Here complex parameters  $\gamma$  and  $\epsilon$  give size of particles  $\left( \frac{|\gamma|}{2\sqrt{|\epsilon|}} \right)$  and distance between them  $\left( 2\sqrt{|\epsilon|} \right)$ .

### 3.1. Perturbation theory

Here we look for correction to ansatz (3.1) by perturbation theory assuming coupling  $\varepsilon$  to be small. To simplify calculations let's assume parameters  $\gamma, \epsilon \in \Re$ , this corresponds to poles lying on axis  $y$ . Equation for the first correction looks like

$$\begin{aligned} & -\frac{32\gamma^3 e^{i\alpha't} (x^2 + y^2) (-\gamma^2 + \zeta)}{\zeta^3 (\epsilon + (x + iy)^2)} - \\ & -\frac{8\gamma e^{-i\alpha't} (\gamma^2 + (\epsilon + (x + iy)^2) (\epsilon - 3(x - iy)^2))}{(\epsilon + (x - iy)^2)^2 \zeta} \\ & + \frac{8\gamma^3 e^{3i\alpha't} (\gamma^2 + (\epsilon - 3(x + iy)^2) (\epsilon + (x - iy)^2))}{\zeta (\epsilon + (x + iy)^2)^4} - \\ & -\alpha'^2 \left( 1 - \frac{3\gamma^6 + 5\gamma^4\zeta + \gamma^2\zeta^2}{\zeta^3} \right) u - \frac{(\gamma^2 + \zeta)^3}{\zeta^3} \partial_t^2 u + \frac{4i\alpha'\gamma^2 (\gamma^2 + \zeta)^2}{\zeta^3} \partial_t u + \\ & + \frac{(\gamma^2 + \zeta)^3}{\zeta^3} (\partial_y^2 u + \partial_x^2 u) + \frac{8\gamma^2(x + iy) (\gamma^2 + \zeta)^2}{\zeta^3 (\epsilon + (x + iy)^2)} (\partial_x u + i\partial_y u) = 0, \end{aligned} \quad (3.2)$$

where

$$\zeta = |(x + iy)^2 + \epsilon|^2 = \frac{\gamma^2}{u_0 u_0^*}. \quad (3.3)$$

In case of distant particles ( $\gamma \ll \epsilon$ ) one can expand the equation in powers of  $\frac{\gamma^2}{\zeta}$ , that in the leading order leads to

$$-\frac{8\gamma e^{-i\alpha't} (\epsilon - 3(x - iy)^2)}{(\epsilon + (x - iy)^2)^3} - \alpha'^2 u - \partial_t^2 u + (\partial_y^2 u + \partial_x^2 u) = 0 \quad (3.4)$$

and gives asymptotes of the first correction at the spatial infinity, i.e. far from lumps:

$$u_1(t, \bar{z}) = -\frac{2i\gamma e^{-i\alpha't} (2\alpha't - i) (\epsilon - 3\bar{z}^2)}{\alpha'^2 (\epsilon + \bar{z}^2)^3}. \quad (3.5)$$

It is necessary to notice some peculiarities of the correction found: firstly, it breaks analyticity of the solution; secondly, the phase rotates in the opposite direction; and finally one can see linear growth with time, that means size increase.

### 3.2. Moduli approximation

Now we say parameters (moduli)  $\gamma = G(t) e^{i\phi(t)}$ ,  $\epsilon = E(t) e^{i\theta(t)}$  of ansatz (3.1) slowly depend on time. Then effective action for such a four-dimensional dynamical system after integrating over  $x, y$  takes the following form (here and below we denote  $\alpha'$  just as  $\alpha$ ; moreover in the very ansatz we include rotation with constant velocity  $\alpha'$  into the definition of  $\phi(t)$ ):

$$\begin{aligned} S = \int dt & \left( -4\pi + f(E, G) \left( \dot{G}^2 + G^2(\dot{\phi}^2 - \alpha^2) \right) + \right. \\ & \left. + g(E, G) \left( \dot{E}^2 + E^2\dot{\theta}^2 \right) + 2h(E, G) \left( \dot{G}\dot{E} + EG\dot{\theta}\dot{\phi} \right) - \frac{\varepsilon}{2} V_{so} \right), \end{aligned} \quad (3.6)$$

where

$$f(E, G) = \frac{\pi}{2\sqrt{E^2 + G^2}} (2K(k) - J(k)), \quad (3.7)$$

$$g(E, G) = \frac{\pi}{2\sqrt{E^2 + G^2}} (J(k)), \quad (3.8)$$

$$h(E, G) = \frac{\pi}{2\sqrt{E^2 + G^2}} \frac{G}{E} (J(k) - K(k)), \quad (3.9)$$

$$k = \frac{E}{\sqrt{E^2 + G^2}}, \quad (3.10)$$

$$V_{so} = \int_0^\infty d\xi \int_0^{2\pi} d\psi 32(1 - k^2)\xi \left( \frac{\xi^2 \cos(2\psi - \phi)}{(1 + \xi^2 + 2k\xi \cos(2\psi - \theta))^2} + \frac{2k\xi \cos(\psi + \theta - \phi) + k^2 \cos(\psi + \phi - 2\theta)}{(1 + \xi^2 + 2k\xi \cos(2\psi - \theta))^2} \right)^2, \quad (3.11)$$

and functions  $K(k)$ ,  $J(k)$  are complete elliptical integrals of the first and of the second kind respectively. This way we obtain explicit form of motion equations:

$$\ddot{G} = \frac{1}{2\Delta} \left( \frac{\dot{G}^2}{G} (f_4 + f_3 - f_2) + f_2 \left( \frac{2\dot{E}\dot{G}}{E} - \frac{G\dot{E}^2}{E^2} + G\dot{\theta}(\dot{\theta} - 2\dot{\phi}) \right) + f_1 G (\dot{\phi}^2 - \alpha^2) \right) + \frac{E^2 + G^2}{2\Delta} ((K(k) - J(k))EG\mathbf{I}_2 + J(k)E^2\mathbf{I}_1), \quad (3.12)$$

$$\ddot{E} = \frac{1}{2\Delta} \left( \frac{\dot{E}^2}{E} (f_1 + f_3 - f_2) + f_3 \left( \frac{E\dot{G}^2}{G^2} - \frac{2\dot{E}\dot{G}}{G} + E(2\dot{\theta}\dot{\phi} + \alpha^2 - \dot{\phi}^2) \right) + f_4 E\dot{\theta}^2 \right) + \frac{E^2 + G^2}{2\Delta} ((2K(k) - J(k))E^2\mathbf{I}_2 + (K(k) - J(k))EG\mathbf{I}_1), \quad (3.13)$$

$$\ddot{\theta} = \frac{1}{\Delta} \left( f_3 \left( \frac{\dot{G}}{G} (\dot{\phi} - \dot{\theta}) - \frac{\dot{E}\dot{\phi}}{E} \right) - f_4 \frac{\dot{E}\dot{\theta}}{E} \right) + \frac{E^2 + G^2}{2\Delta} ((K(k) - J(k))\mathbf{I}_4 + (2K(k) - J(k))\mathbf{I}_3), \quad (3.14)$$

$$\ddot{\phi} = \frac{1}{\Delta} \left( f_2 \left( \frac{\dot{E}}{E} (\dot{\phi} - \dot{\theta}) + \frac{\dot{G}\dot{\theta}}{G} \right) - f_1 \frac{\dot{G}\dot{\phi}}{G} \right) + \frac{E^2 + G^2}{2\Delta} \left( J(k) \frac{E^2}{G^2} \mathbf{I}_4 + (K(k) - J(k))\mathbf{I}_3 \right), \quad (3.15)$$

where

$$f_1 = J^2(k) (E^2 + G^2) (4E^2 + G^2) - 2J(k)K(k) (2E^4 + 4G^2E^2 + G^4) + K^2(k) (2E^2 + G^2) G^2, \quad (3.16)$$

$$f_2 = 3J^2(k) (E^2 + G^2) E^2 - 2J(k)K(k) (E^2 + 2G^2) E^2 + K^2(k)G^2E^2, \quad (3.17)$$

$$f_3 = J^2(k) (E^2 + G^2) G^2 - 2J(k)K(k)G^4 + K^2(k)G^4, \quad (3.18)$$

$$f_4 = J^2(k) (E^2 + G^2) E^2 - 2J(k)K(k) (E^2 + 2G^2) E^2 + K^2(k)E^2G^2, \quad (3.19)$$

$$\Delta = f_3 + f_4, \quad (3.20)$$

$$\mathbf{I}_1 = \frac{\varepsilon}{\pi} \sqrt{E^2 + G^2} \frac{\partial V_{so}}{\partial G} = \frac{\varepsilon}{\pi} \sqrt{E^2 + G^2} \frac{\partial V_{so}}{\partial k} \frac{\partial k}{\partial G}, \quad (3.21)$$

$$\mathbf{I}_2 = \frac{\varepsilon}{\pi} \sqrt{E^2 + G^2} \frac{\partial V_{so}}{\partial E} = \frac{\varepsilon}{\pi} \sqrt{E^2 + G^2} \frac{\partial V_{so}}{\partial k} \frac{\partial k}{\partial E}, \quad (3.22)$$

$$\mathbf{I}_3 = \frac{\varepsilon}{\pi} \sqrt{E^2 + G^2} \frac{\partial V_{so}}{\partial \theta}, \quad (3.23)$$

$$\mathbf{I}_4 = \frac{\varepsilon}{\pi} \sqrt{E^2 + G^2} \frac{\partial V_{so}}{\partial \phi}. \quad (3.24)$$

Equations (3.12)–(3.15) can be solved numerically. Then  $\gamma(0), \epsilon(0)$  give initial configuration,  $\dot{\gamma}(0), \dot{\epsilon}(0)$  — a small perturbation. In our case  $\dot{G}, \dot{E}, \dot{\theta} = 0, \dot{\phi} = \alpha$ . Equations were solved using Runge-Kutta 4th order method for following parameter values: couplings  $\alpha = 0.03, \varepsilon = 0.1$ , lumps' size  $\frac{G}{2\sqrt{E}} = 1$ , distance between lumps  $2\sqrt{E} = 50$ . Integrals  $\mathbf{I}_1$ – $\mathbf{I}_4$  were also calculated numerically.

#### 4. Results

Several effects of spin-orbit interaction on two-particle solution were discovered.

1. Size increase: this effect is in good correspondence to the growth found with perturbation theory.

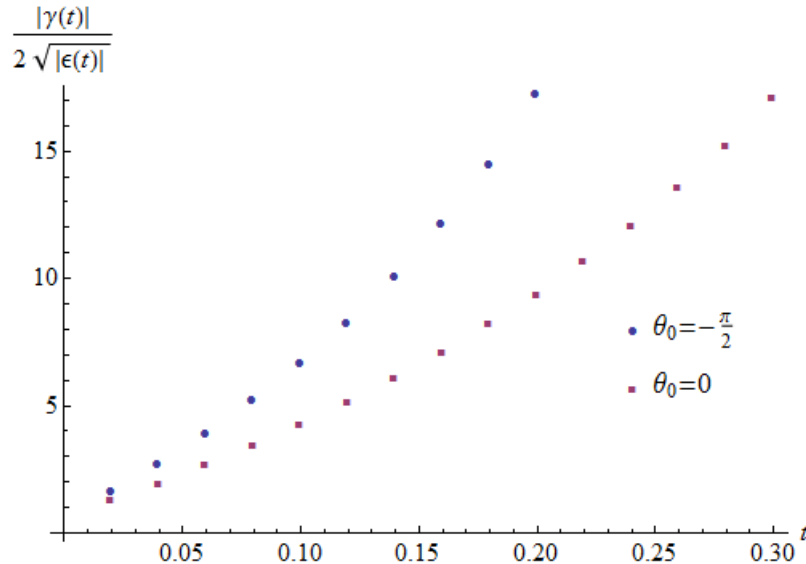


Figure 4.1: Particle size evolution



2. Attraction: distance between particles declines, that was not seen from correction (3.5). The fact, however, is due to expression (3.5) gives only asymptotes at spatial infinity, which remain the same.

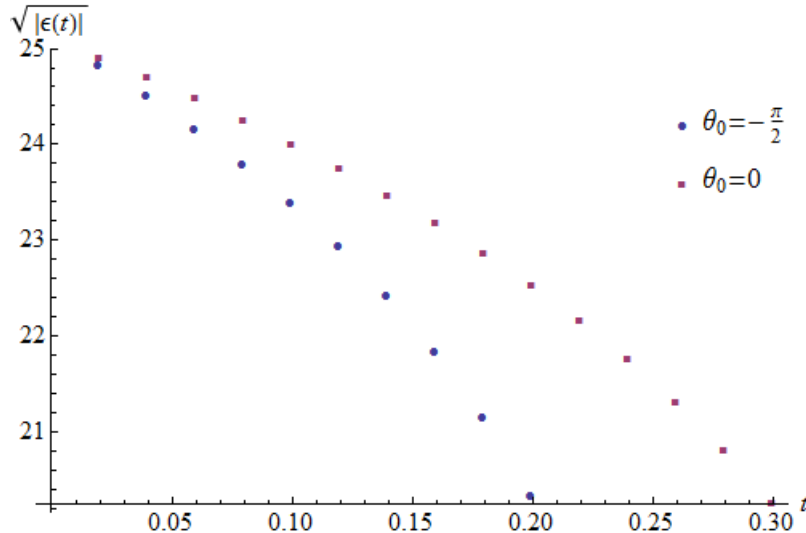


Figure 4.2: Evolution of distance between particle

3. Dependence on phases: figures 4.1, 4.2 show numerical solutions for two choice of parameter  $\theta_0 = -\frac{\pi}{2}$  and  $\theta_0 = 0$  that determine initial positions of Q-lumps. The first one corresponds to particles lying on the real axis, the second one — on the imaginary one. While system behaviour remains qualitatively the same, one can notice that velocities of processes depend significantly on the parameter. Thus in contrast to functions (3.7)-(3.9) not depending on phases  $\theta$ ,  $\phi$  and allowing to consider just functions  $G(t)$ ,  $E(t)$  in absence of spin-orbit interaction (in case of the same initial  $\dot{\theta}(0) = 0$ ,  $\dot{\phi}(0) = \alpha$ ), addition of  $\mathcal{L}_{so}$  results in appearance of such a dependence. In Fig. 4.3 one can see how potential  $V_{so}(k)$  depends on phases: indeed for every fixed  $\theta$ ,  $\phi$  profile  $V_{so}(k)$  qualitatively remains the same and results in growth and attraction of particles.

## 5. Conclusion

In this paper we considered  $O(3)$   $\sigma$ -model with an additional potential term on domain wall in 3+1-dimensional spacetime admitting solitonic solutions (Q-lumps). Effects caused by spin-orbit interaction in the bulk on two-particle solutions were studied.

Asymptotic behaviour at spatial infinity of first corrections to two-particle ansatz was found perturbatively; evolution of initially motionless configuration was numerically studied through adiabatic approximation. We discovered that in presence of spin-orbit interaction Q-lumps begin to interact: they grow and attract; also entanglement between internal degrees of freedom and coordinates leads to dependence on phases of configuration.

One important feature of Q-lumps in (2+1) discovered by Leese was that the additional term, which significantly change behaviour of the system, can be treated as just external perturbation. The interaction effects outlined in sect. 4 suggest that another perturbation to the action in media (in bulk) can "break" some properties of Leese's non-interacting lumps living on the (2+1) dimensional wall, and all this happens even when taking small both couplings.

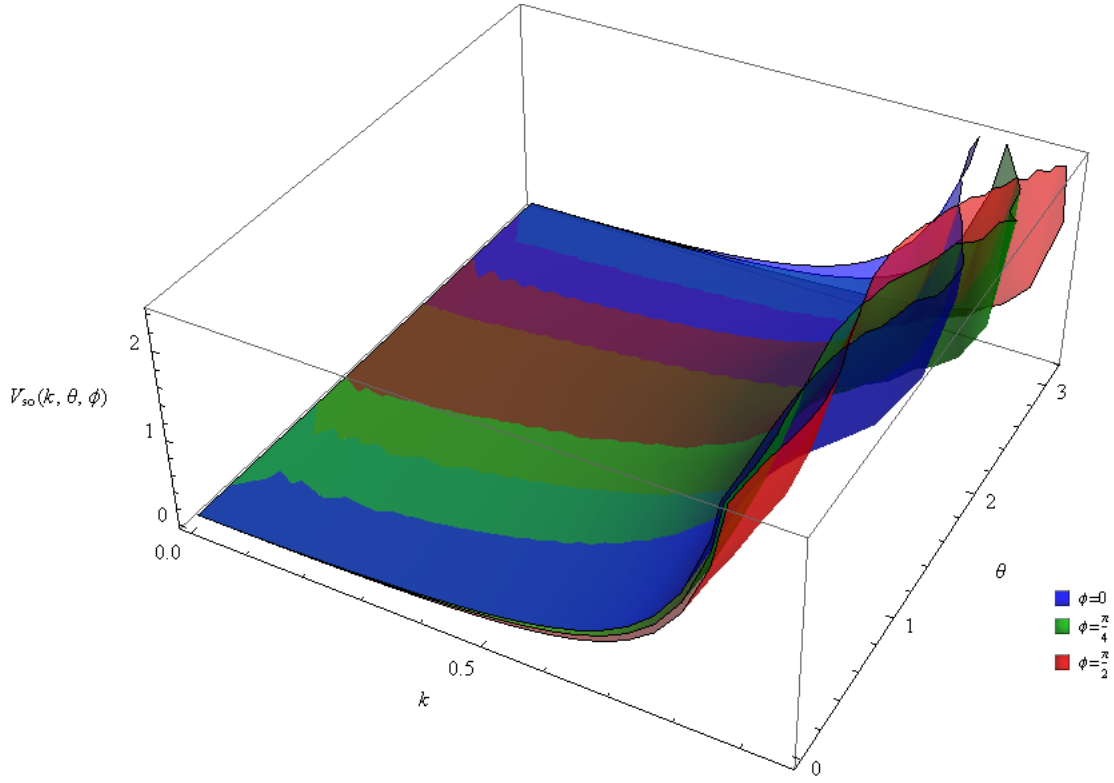


Figure 4.3: Spin-orbit interaction potential  $V_{so}(k, \theta, \phi)$

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